

A Qualitative Analysis of A Bianchi Type IV Viscous Fluid Model

Ikjyot Singh Kohli*

York University - Department of Physics and Astronomy

Michael C. Haslam†

York University - Department of Mathematics and Statistics

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A Bianchi Type IV viscous fluid model is analyzed for asymptotic isotropy behaviour. In particular, we consider the case of a viscous fluid without heat conduction in conjunction with a non-tilted Bianchi cosmology in an orthonormal frame using expansion-normalized variables. The Einstein field equations in this approach reduce to an autonomous system of first-order ordinary differential equations which because of their high degree of non-linearity are solved using numerical Runge-Kutta methods. It is shown that for cases where the expansion-normalized bulk viscosity coefficient ξ_0 dominates over the expansion-normalized shear viscosity coefficient, η_0 , the model isotropizes.

I. INTRODUCTION

Spatially homogeneous and anisotropic models of the universe have undergone great study and continue to remain amongst the most popular areas of research in cosmology. Early-universe cosmological models for the most part have assumed the universe to be spatially homogeneous and anisotropic, with the important exception being the case of inhomogeneous models such as the LeMaitre-Tolman-Bondi and “Swiss-cheese” models. However, if one begins with the idea of the early universe being spatially homogeneous and anisotropic, then to transition to the present-day Friedmann-LeMaitre-Robertson-Walker models requires the anisotropy in the former models to decay. The process by which this anisotropic decay occurs is arguably the most fundamental property of any early-universe model which aims to transition to the present-day models. For example, Belinskii, Khalatnikov, and Lifshitz [1] studied the oscillatory approach to a singular point in relativistic cosmologies. Misner [2] studied the anisotropic decay of the vacuum Bianchi Type IX/Mixmaster models of the universe. A very general approach to describing the isotropization of Bianchi models was described by Salucci and Fabbri [3]. Coley and van den Hoogen [4] studied causal anisotropic viscous fluid models and described conditions for such models to isotropize. As for very recent work on this subject, Pradhan, Rai, and Singh [5] studied the Bianchi Type V bulk viscous models and showed that such models do isotropize for specific functional forms of the anisotropic scale factors.

As for Bianchi Type IV models specifically, Hervik, van den Hoogen, and Coley [6] studied future asymptotic behaviour of tilted vacuum Bianchi Type IV models, and found that such models do not necessarily isotropize at late times. Uggla and Rosquist [7] studied the orthogonal Bianchi Type IV model near the initial singularity with a vacuum or perfect-fluid source. We chose to study the isotropization behaviour of the Bianchi Type IV viscous model largely because such a study has not been taken on extensively in the literature, and perhaps, such a study will add to the already rich landscape of spatially homogeneous and anisotropic models of the early universe. We will employ the important technique of *orthonormal frames* pioneered by Ellis and MacCallum [8] which reduces the Einstein field equations which are a coupled set of ten hyperbolic nonlinear partial differential equations into a system of autonomous nonlinear ordinary differential equations.

II. THE ORTHONORMAL FRAME APPROACH TO THE BIANCHI TYPE IV ALGEBRA

If one considers a fluid described by a family of time-like curves, with 4-velocity, u^a , the properties of the fluid flow are most appropriately described by the decomposition [9]

$$u_{a;b} = -a_a u_b + \sigma_{ab} + \omega_{ab} + \frac{1}{3} h_{ab} \theta. \quad (1)$$

*Electronic address: isk@yorku.ca

†Electronic address: mchaslam@mathstat.yorku.ca

In this equation $\theta \equiv u_{;a}^a$ is a measure of the divergence of the family of time-like curves, $a_a \equiv u_{a;b}u^b$ is a measure of how much the curves represent non-geodesics (and thus, can be taken to be the fluid acceleration), and $h_{ab} \equiv g_{ab} - u_a u_b / u^c u_c$ is a projection tensor. We also have terms that represent the shear and vorticity of the fluid:

$$\sigma_{ab} = \left[\frac{1}{2} (u_{m;n} + u_{n;m}) - \frac{1}{3} u_{;c}^c h_{mn} \right] h_a^m h_b^n; \quad (2)$$

$$\omega_{ab} = \frac{1}{2} (u_{m;n} - u_{n;m}) h_a^m h_b^n. \quad (3)$$

In addition, we note that since for any velocity field $u^a u_a = \pm 1$, the following relationships also hold:

$$a_c u^c = \omega_{nc} u^c = \sigma_{nc} u^c = h_{nc} u^c = 0. \quad (4)$$

When using orthonormal frames, the metric tensor, g_{uv} takes a very simple form for the following reason: any cosmological model can be described by a coordinate system in which we can define a basis $\{\mathbf{e}_u\}$, and its dual basis of differential one-forms $\{\omega^u\}$. As usual, the metric tensor is then defined as $g_{uv} = \mathbf{g}(\mathbf{e}_u, \mathbf{e}_v)$, from which the line element is $ds^2 = g_{uv} \omega^u \omega^v$. For a given coordinate chart on the pseudo-Riemannian manifold, we can take the basis formed by $\{\mathbf{e}_u\}$ as the *coordinate basis* $\{\partial/\partial x^i\}$, with the dual basis then playing the role of the coordinate differential one-forms, $\{dx^i\}$. If we now assume that the basis constitutes an *orthonormal frame* \mathfrak{o} , then all four basis vectors $\{\mathbf{e}_u\}$ are mutually orthonormal, and thus satisfy the relationship: $\mathbf{g}(\mathbf{e}_u, \mathbf{e}_v) = \eta_{uv} = \delta_{uv} = \text{diag}(-1, 1, 1, 1)$. Hence, g_{uv} is really just the Kronecker tensor, and all tensors in this approach have their indices lowered(raised) with it. Furthermore, the group structure constants are functions, which we denote by a lowercase c to distinguish between the constants typically described in the non-orthonormal frame approach by an uppercase C [10] [11]. These functions satisfy the commutation relation [12] [13]

$$[\mathbf{e}_u, \mathbf{e}_v] = c_{uv}^a \mathbf{e}_a. \quad (5)$$

Recall that for an arbitrary basis, for a general (torsion-free) spacetime, we have

$$c_{uv}^a = \Gamma_{vu}^a - \Gamma_{uv}^a. \quad (6)$$

We can then write the connection coefficients as

$$\Gamma_{auv} = \frac{1}{2} (g_{ab} c_{vu}^b + g_{uv} c_{av}^b - g_{vb} c_{au}^b). \quad (7)$$

Because we are assuming a non-tilted cosmology, the fluid four-velocity is necessarily orthogonal to the spatial hypersurfaces, which implies that the fluid is irrotational and geodesic. That is,

$$\omega_{uv} = u_{u;v} u^v = 0. \quad (8)$$

We can therefore write that

$$\theta_{uv} = u_{u;v} = \frac{1}{3} \theta h_{uv} + \sigma_{uv}. \quad (9)$$

Exploiting the non-tilted condition once more, we see that Eq. (9) implies

$$\theta_{uv} = \Gamma_{uv}^t \Rightarrow c_{ta}^t = c_{ab}^t = 0. \quad (10)$$

Using the Behr decomposition, the structure coefficients(functions) can be decomposed into symmetric and anti-symmetric parts as

$$c_{tb}^a = -\theta_b^a + \epsilon_{bc}^a \Omega^c. \quad (11)$$

The vector Ω^c has the significance of being interpreted as a local angular velocity in the rest frame of an observer with spatial frame $\{\mathbf{e}_a\}$. It is conventionally defined as [12]

$$\Omega^a = \frac{1}{2} \epsilon^{abcd} u_b \mathbf{e}_c \cdot \dot{\mathbf{e}}_d. \quad (12)$$

It can be shown that the structure coefficients are spatial, and therefore, must correspond to the Bianchi models. We can therefore write that

$$c_{ij}^k = \epsilon_{ijl} n^{lk} + a_l (\delta_i^l \delta_j^k - \delta_j^l \delta_i^k). \quad (13)$$

Since each structure coefficient is constant along each orbit of transitivity, n^{ab} and a_i are functions of time only. We can find evolution equations for n^{ab} and a_i by first realizing that the Jacobi identity holds for all vectors. The Jacobi identity as applied to the set of vectors $(\mathbf{U}, \mathbf{e}_a, \mathbf{e}_b)$ gives

$$\begin{aligned} & [\mathbf{U}, [\mathbf{e}_a, \mathbf{e}_b]] + [\mathbf{e}_a, [\mathbf{e}_b, \mathbf{U}]] + [\mathbf{e}_b, [\mathbf{U}, \mathbf{e}_a]] = 0 \\ \Rightarrow & (\mathbf{U}(c_{ab}^v) + c_{tu}^v c_{ab}^u + c_{au}^v c_{bt}^u + c_{bu}^v c_{ta}^u) \mathbf{e}_v = 0. \end{aligned} \quad (14)$$

In this derivation, \mathbf{U} is a gauge vector that will typically be set to ∂_t . Note that since we are assuming that the fluid is non-tilted we necessarily have $\theta_{uv} = \Gamma_{uv}^t \Rightarrow c_{ta}^t = c_{ab}^t = 0$. We therefore obtain the identity

$$\partial_t(c_{ab}^k) + c_{td}^k c_{ab}^d + c_{ad}^k c_{bt}^d + c_{bd}^k c_{ta}^d = 0. \quad (15)$$

One can show that upon applying Jacobi's identity (Eq. (14)) to the three spatial vectors, we get the eigenvalue equation [12]

$$n^{ij} a_i = 0. \quad (16)$$

The trace of Eq. (15) gives the evolution equation for a_i as

$$\dot{a}_i + \frac{1}{3} \theta a_i + \sigma_{ij} a^j + \epsilon_{ijk} a^j \Omega^k = 0. \quad (17)$$

On the other hand, the trace-free part of Eq.(15) gives the evolution equation for n_{ab} as

$$n_{ab} + \frac{1}{3} \theta n_{ab} + 2n_{(a}^k \epsilon_{b)kl} \Omega^l - 2n_{k(a} \sigma_{b)}^k = 0. \quad (18)$$

The structure constants as defined in Eq. (13) must correspond to a Lie algebra as per the Jacobi identity (Eq. (14)), thus, Eq. (16) must hold for any choice of n_{ij} and a_i . Bianchi models are typically classified into two categories: Class A models and Class B models. Class A models are all Bianchi models such that $a_i = 0$ for which Eq. (16) is satisfied. For class B models, a_i must be an eigenvector of n^{ij} with zero eigenvalue. It is important to note that we always take n^{ij} to be a symmetric matrix and as such we can diagonalize it using an orientation of our choice for the spatial frame. It has been conventional to assume

$$n_{ij} = \text{diag}(n_1, n_2, n_3), \quad a^i = (0, 0, a). \quad (19)$$

The Jacobi identity (Eq. (14)) immediately implies that $n_{33}a = n_3a = 0$. It is a fact of linear algebra that eigenvalues of a matrix are invariant under a conjugation operation with respect to rotations. One can then classify the different Bianchi models by the signs of the eigenvalues $n_{11}, n_{22}, n_{33} = n_1, n_2, n_3$ and a . These have been listed in Table 1. Since, we are interested in the Bianchi Type IV algebra, we choose

Type	a	n_1	n_2	n_3
I	0	0	0	0
II	0	+	0	0
VI_0	0	+	-	0
VII_0	0	+	+	0
$VIII$	0	+	+	-
IX	0	+	+	+
V	+	0	0	0
IV	+	+	0	0
VI_h	+	+	-	0
VII_h	+	+	+	0

TABLE I: Bianchi Type classifications in terms of the *time-dependent* eigenvalues.

$$a^i = a \delta_3^i > 0, \text{ and } n_1 > 0, n_2 = n_3 = 0. \quad (20)$$

III. THE ENERGY-MOMENTUM TENSOR FOR A VISCOUS FLUID

In this section, we will derive the form of the energy-momentum tensor under concern, namely, for that of a viscous fluid without heat conduction. We will then derive the dynamical equation for this fluid, and will see in the next section that the latter closes the system of dynamical equations. Recall that the energy-momentum tensor for a perfect fluid takes the form

$$T^{ab} = (\mu + p)u^a u^b - u^c u_c g^{ab} p. \quad (21)$$

For the moment, letting $\mu + p = w$, we obtain

$$T^{ab} = w u^a u^b - u^c u_c g^{ab} p, \quad (22)$$

Denoting the viscous contributions by \mathcal{V}_{ab} , we seek a modification of Eq. (22) such that

$$T_{ab} = w u_a u_b - u_c u^c g_{ab} p + \mathcal{V}_{ab}. \quad (23)$$

To obtain the form of this additional tensor term, we note that from classical fluid mechanics, the Euler equation is given as

$$(\rho u_i)_{,t} = -\Pi_{ik,k}, \quad (24)$$

where Π_{ik} is the momentum flux tensor. Also, recall that for a non-viscous fluid, one has the fundamental relationship

$$\Pi_{ik} = p\delta_{ik} + \rho u_i u_k. \quad (25)$$

We simply add a term to Eq. (25) that represents the viscous momentum flux, Σ'_{ik} , to obtain

$$\Pi_{ik} = p\delta_{ik} + \rho u_i u_k - \Sigma'_{ik} = -\Sigma_{ik} + \rho u_i u_k. \quad (26)$$

It is important to note that

$$\Sigma_{ik} = -p\delta_{ik} + \Sigma'_{ik} \quad (27)$$

is the stress tensor, while, Σ'_{ik} is the *viscous* stress tensor. The general form of the viscous stress tensor can be formed by recalling that viscosity is formed when the fluid particles move with respect to each other at different velocities, so this stress tensor can only depend on spatial components of the fluid velocity. We assume that these gradients in the velocity are small, so that the momentum tensor only depends on the first derivatives of the velocity in some Taylor series expansion. Therefore, Σ'_{ik} is some function of the $u_{i,k}$. In addition, when the fluid is in rotation, no internal motions of particles can be occurring, so we consider linear combinations of $u_{i,k} + u_{k,i}$, which clearly vanish for a fluid in rotation with some angular velocity, Ω_i . The most general viscous tensor that can be formed is given by

$$\Sigma'_{ik} = \eta \left(u_{i,k} + u_{k,i} - \frac{2}{3} \delta_{ik} u_{l,l} \right) + \xi \delta_{ik} u_{l,l}, \quad (28)$$

where η and ξ are the coefficients of shear and bulk/second viscosity, respectively [14] [15]. In Eq. (28), we note that $\delta_{ik} u_{l,l}$ is an expansion rate tensor, and $(u_{i,k} + u_{k,i} - \frac{2}{3} \delta_{ik} u_{l,l})$ represents the shear rate tensor. Since we would like to generalize this expression to the general relativistic case, we replace the partial derivatives above with covariant derivatives, and the Kronecker tensor with a more general metric tensor, that is, $\delta_{ik} \rightarrow g_{ik}$. We thus have that

$$\Sigma'_{ik} = \eta \left(u_{i;k} + u_{k;i} - \frac{2}{3} g_{ik} u_{l;l} \right) + \xi g_{ik} u_{l;l}. \quad (29)$$

Denoting the shear rate tensor as σ_{ab} , and the expansion rate scalar as $\theta \equiv u^a_{;a}$, Eq. (29) becomes

$$\mathcal{V}_{ab} = -2\eta\sigma_{ab} - \xi\theta h_{ab}. \quad (30)$$

Substituting Eq. (30) into Eq. (23) we finally obtain the required form of the energy-momentum tensor as

$$T_{ab} = (\mu + p) u_a u_b - u_c u^c g_{ab} p - 2\eta\sigma_{ab} - \xi\theta h_{ab}. \quad (31)$$

A. An Equation of Motion for the Fluid

Now that we have the energy-momentum tensor in hand (Eq. (31)), we are in a position to derive an equation of motion for the fluid. Equations of motion are typically found from conservation laws. The generalization of conservation of energy-momentum in general relativity appears via the divergence-free property of the energy-momentum tensor, which itself, is coupled to the divergence-free property of the Einstein tensor. For notational convenience (which will be clearer in what follows), we will let $\pi_{ab} = 2\eta\sigma_{ab}$, such that

$$T_a^b = (\mu + p)u_a u^b + \delta_a^b p - \pi_a^b - \xi\theta (\delta_a^b + u_a u^b). \quad (32)$$

Energy conservation requires that this quantity should have a vanishing divergence, $T_{;b}^{ab} = 0 = T_{a;b}^b$. Taking the divergence of Eq. (32) and contracting the resulting equation with u^a , we obtain

$$u^a ((\mu + p)u_a u^b + \delta_a^b p - \pi_a^b - \xi\theta (\delta_a^b + u_a u^b))_{;b} = 0. \quad (33)$$

We will proceed by evaluating Eq. (33) term-by-term. For the first term, we obtain

$$\begin{aligned} u^a ((\mu + p)_{;b} u_a u^b + (\mu + p)u_{a;b} u^b + (\mu + p)u_a u_{;b}^b) \\ = (-\dot{\mu} - \dot{p} - (\mu + p)\theta). \end{aligned} \quad (34)$$

For the second term, we obtain

$$\begin{aligned} u^a (\delta_a^b p)_{;b} \\ = \dot{p}. \end{aligned} \quad (35)$$

For the third term, we have

$$\begin{aligned} u^a (\pi_a^b)_{;b} \\ = -\sigma_{ab}\pi^{ab}. \end{aligned} \quad (36)$$

For the fourth term, we have

$$\begin{aligned} u^a (\xi\theta (\delta_a^b + u_a u^b))_{;b} \\ = -\xi\theta^2. \end{aligned} \quad (37)$$

Substituting Eqs. (34) - (37) into Eq. (33), we obtain

$$\dot{\mu} + (\mu + p)\theta - 2\eta\sigma_{ab}\sigma^{ab} - \xi\theta^2 = 0. \quad (38)$$

Using the definition $\sigma^2 = \frac{1}{2}\sigma^{ab}\sigma_{ab} \Rightarrow \sigma^{ab}\sigma_{ab} = 2\sigma^2$, we have finally that

$$\dot{\mu} + (\mu + p)\theta - 4\eta\sigma^2 - \xi\theta^2 = 0. \quad (39)$$

Before we conclude this section, we note the necessity of defining a barotropic equation of state. We will choose $p = \frac{1}{3}\mu$, which comes from the idea of the fluid moving ultra-relativistically as would be the case in the early universe. Therefore, substituting this equation of state into Eq. (39), we obtain

$$\dot{\mu} + \left(\frac{4}{3}\mu\right)\theta - 4\eta\sigma^2 - \xi\theta^2 = 0. \quad (40)$$

This is the evolution equation for the fluid as it propagates through the universe. As we shall see in the next section, this is the dynamical equation which closes the Einstein field equations.

IV. THE DYNAMICAL EQUATIONS

Using the orthonormal frame approach we will now write the Einstein field equations that correspond to the Bianchi Type IV algebra. The resulting equations are the Raychaudhuri equation, the generalized Friedmann equation, and shear propagation equations. The system will be closed by Eqs. (17), (18) and (40).

A. The Raychaudhuri Equation

We now derive the viscous analogue of the Raychaudhuri equation which describes the expansion/contraction behaviour of a congruence of fluid lines, and necessarily the universe. We begin by recalling that one can consider a fluid evolution as described by a family of time-like curves, with 4-velocity, u^a . We had from Eq. (1) that

$$u_{a;b} = -a_a u_b + \sigma_{ab} + \omega_{ab} + \frac{1}{3}h_{ab}\theta. \quad (41)$$

Analyzing the equation of the divergence of the fluid flow, we are interested in how this divergence evolves in time. That is, we are interested in the quantity

$$\dot{\theta} = (u^a_{;a})_{;b} u^b = u^a_{;a;b} u^b. \quad (42)$$

By the definition of the Riemann curvature tensor, we have that

$$u^a_{;a;b} - u^a_{;b;a} = -R_{bc}u^c. \quad (43)$$

Substituting Eq. (43) into Eq. (42), we obtain

$$\dot{\theta} = u^a_{;b;a} u^b - R_{bc}u^c u^b. \quad (44)$$

Using the identity $(u^a_{;b} u^b)_{;a} = u^a_{;b;a} u^b + u^a_{;b} u^b_{;a}$, Eq. (44) becomes

$$\dot{\theta} = a^a_{;a} - R_{bc}u^c u^b - u^a_{;b} u^b_{;a}. \quad (45)$$

Since $h_{ab}h^{ab} = 3$, $\sigma^2 = \frac{1}{2}\sigma^{ab}\sigma_{ab}$, $\omega^2 = \frac{1}{2}\omega^{ab}\omega_{ab}$, $h^{ab}\sigma_{ab} = 0$, and

$$h^{ab}u_{a;b} = -a_a u_b h^{ab} + h^{ab}\sigma_{ab} + h^{ab}\omega_{ab} + \frac{1}{3}\theta h_{ab}h^{ab}, \quad (46)$$

one obtains

$$\dot{\theta} = -R_{bc}u^b u^c + (a^a_{;a} - \Gamma^a_{an}a_n) + 2\omega^2 - 2\sigma^2 - \frac{1}{3}\theta^2. \quad (47)$$

The Einstein field equations imply that

$$R_{ab} = \kappa \left(T_{ab} - \frac{1}{2}Tg_{ab} \right). \quad (48)$$

Computing the trace of the energy-momentum tensor (Eq. (32)) we obtain

$$T = u^a u_a (\mu - 3p) - 2\eta g^{ab}\sigma_{ab} - \xi g^{ab}(\theta h_{ab}). \quad (49)$$

Contracting Eq. (2) with the metric tensor g^{ab} we get that

$$g^{ab}\sigma_{ab} = g^{ab} \left(u_{(a;b)} + a_{(a}u_{b)} - \frac{1}{3}\theta h_{ab} \right) = 0. \quad (50)$$

In addition, note that the contraction of θh_{ab} with the metric tensor g_{ab} simply gives $\theta g^{ab}h_{ab} = 3\theta$. Therefore, Eq. (49) now takes the simple form

$$T = u^a u_a (\mu - 3p) - 3\xi\theta. \quad (51)$$

We also make use of the additional property, namely that

$$T_{ab}u^a u^b = \mu. \quad (52)$$

Substituting Eqs. (49) - (52) into Eq. (48) we get

$$R_{ab}u^a u^b = \frac{1}{2}\kappa(\mu + 3p) + \frac{3}{2}\kappa\xi\theta u^a u_a. \quad (53)$$

We will use the metric signature $(-1, +1, +1, +1)$ so that $u^a u_a = -1$, and Eq. (53) becomes

$$R_{ab} u^a u^b = \frac{1}{2} \kappa (\mu + 3p) - \frac{3}{2} \kappa \xi \theta. \quad (54)$$

Substituting Eq. (54) into Eq. (47) we obtain the *Raychaudhuri* equation for our viscous fluid as

$$\dot{\theta} = -\frac{1}{2} \kappa (\mu + 3p) + \frac{3}{2} \kappa \xi \theta + a_{;a}^a + 2\omega^2 - 2\sigma^2 - \frac{1}{3} \theta^2. \quad (55)$$

Since we are assuming a non-tilted cosmology, the fluid is necessarily geodesic and irrotational, so that Eq. (55) upon applying the ultra relativistic equation of state becomes

$$\dot{\theta} = -\kappa \mu + \frac{3}{2} \kappa \xi \theta - 2\sigma^2 - \frac{1}{3} \theta^2. \quad (56)$$

B. The Generalized Friedmann Equation

The generalized Friedmann equation is an extension of the standard Friedmann equation one obtains from the FLRW metric. This equation is important because it relates the rate of expansion to the shear, and more importantly, the curvature of the spatial slices in the spacetime foliation. Exploiting the assumption of a non-tilted cosmology again, we state the additional fact that in this case

$$u_{a;b} = \theta_{ab} = K_{ab}. \quad (57)$$

Here, we have denoted the *extrinsic curvature* of the spatial slices by the *extrinsic curvature tensor*, K_{ab} . Recall from standard differential geometry, we have that for any three-dimensional spatial slice,

$$\begin{aligned} {}^{(4)}R &= {}^{(3)}R + K^2 - K^{\alpha\beta} K_{\alpha\beta} - 2 {}^{(4)}R_{\alpha\beta} u^\alpha u^\beta \\ \Rightarrow \kappa T_{\alpha\beta} u^\alpha u^\beta &= \frac{1}{2} \left({}^{(3)}R - K^{\alpha\beta} K_{\alpha\beta} + K^2 \right). \end{aligned} \quad (58)$$

The decomposition equation (Eq. (1)) then becomes

$$u_{\alpha;\beta} = \theta_{\alpha\beta} = K_{\alpha\beta} = \sigma_{\alpha\beta} + \frac{1}{3} h_{\alpha\beta} \theta. \quad (59)$$

Substituting the extrinsic curvature scalar as $K^2 \equiv \frac{1}{2} K_{\alpha\beta} K^{\alpha\beta}$, and Eq. (52) into Eq. (58), one obtains the *generalized Friedmann equation*,

$$\frac{1}{3} \theta^2 = \frac{1}{2} \sigma_{ab} \sigma^{ab} - \frac{1}{2} {}^{(3)}R + \kappa \mu. \quad (60)$$

C. The Shear Propagation Equations

The last set of dynamical equations implied by The Einstein Field equations are the *shear propagation equations*. They essentially describe the evolution of the anisotropy in a cosmological model as a function of time. The lengthy derivation of these equations has been done in much of the literature on cosmological dynamical systems [13] [12] [16]. The idea behind the derivation is similar to that of the derivation of the Raychaudhuri equation. The *shear propagation equations* are

$$\dot{\sigma}_{ab} + \theta \sigma_{ab} - \sigma_a^d \epsilon_{bcd} \Omega^c - \sigma_b^d \epsilon_{acd} \Omega^c + {}^{(3)}R_{ab} - \frac{1}{3} \delta_{ab} {}^{(3)}R = 2\eta \kappa \sigma_{ab}. \quad (61)$$

D. The Ricci Curvature and Constraint Equations

Perhaps the greatest accomplishment of the orthonormal frame approach is that the curvature tensors, namely, the Ricci tensor are no longer expressed in terms of the coordinate basis functions which simplifies computations

considerably. The Ricci tensor and scalar are expressed entirely in terms of a^i and n_{ab} . As discussed by Ellis and MacCallum [8] and Grøn and Hervik [12], the Ricci tensor takes the form

$$\begin{aligned} {}^{(3)}R_{ab} = & -\epsilon_a^{cd}n_{bc}a_d - \epsilon_b^{cd}n_{ac}a_d + \\ & 2n_{ad}n_b^d - nn_{ab} - \delta_{ab}\left(2a^2 + n_{cd}n^{cd} - \frac{1}{2}n^2\right). \end{aligned} \quad (62)$$

Contracting Eq. (62), we obtain the Ricci scalar,

$${}^{(3)}R = {}^{(3)}R_a^a = -\left(6a^2 + n_{cd}n^{cd} - \frac{1}{2}n^2\right). \quad (63)$$

One can also show that the off-diagonal components of the four-dimensional Ricci tensor yield a constraint equation,

$$3a^b\sigma_{ba} - \epsilon_{abc}n^{cd}\sigma_d^b = 0. \quad (64)$$

For computational purposes, it is more convenient to express Eq. (64) in component form,

$$\begin{aligned} 3a\sigma_{33} + (n^{11} - n^{22})\sigma_{21} &= 0 \\ 3a\sigma_{31} + n^{22}\sigma_{32} &= 0 \\ 3a\sigma_{32} - n^{11}\sigma_{31} &= 0. \end{aligned} \quad (65)$$

Up to this point we have derived and discussed all of the necessary formalism required to adequately describe the dynamics of a spatially homogeneous and anisotropic spacetime. In the next section, we will apply these results to the Bianchi IV algebra.

V. THE BIANCHI TYPE IV DYNAMICAL EQUATIONS

In this section, we will derive the dynamical equations associated with the Bianchi Type IV algebra, and study their solution. For the Bianchi Type IV geometry, recall from Eq. (20), we take $n_{11} = N$, and $n_{22} = n_{33} = 0$, where $N > 0$. The constraint relations Eq. (65) then imply that

$$\begin{aligned} 3a\sigma_{33} + N\sigma_{21} &= 0 \\ 3a\sigma_{31} &= 0 \\ 3a\sigma_{32} - \sigma_{31} &= 0. \end{aligned} \quad (66)$$

We can see from Eq. (66) that $\sigma_{32} = 0$. Since the shear tensor is taken to be symmetric and traceless, in three dimensions, it only has two independent components. It is also clear from Eq. (66), that the only non-zero off-diagonal component of the shear tensor is $\sigma_{21} = \sigma_{12}$. Therefore, for the case where $a = b$,

$$\sigma_{ab} = \left(\sigma_+ + \sqrt{3}\sigma_-, \sigma_+ - \sqrt{3}\sigma_-, -2\sigma_+\right). \quad (67)$$

The non-zero, independent, off-diagonal component for the shear tensor is given by:

$$\sigma_{21} = \sigma_{12} = \frac{-3a\sigma_{33}}{N} = \frac{6a\sigma_+}{N}. \quad (68)$$

Substituting Eqs. (67), (68), (20) into Eqs. (17) and (18), we obtain the relations,

$$\Omega^1 = \Omega^2 = 0, \Omega^3 = -6a\sigma_+, \quad (69)$$

$$\dot{a} + \frac{1}{3}\theta a - 2\sigma_+a = 0, \quad (70)$$

and

$$\dot{N} - \frac{1}{3}\theta N - 2N\left(\sigma_+ + \sqrt{3}\sigma_-\right) = 0. \quad (71)$$

Using Eqs.(62) and (20), we compute the three-dimensional Ricci tensor as

$${}^{(3)}R_{ab} = -\epsilon_a^{13}n_{b1}(a) - \epsilon_b^{13}n_{a1}(a) + 2n_{a1}n_b^1 - Nn_{ab} - \delta_{ab}\left(2a^2 + \frac{1}{2}N^2\right). \quad (72)$$

Eq. (72) then gives the components of the Ricci tensor as:

$${}^{(3)}R_{11} = \frac{1}{2}N^2 - 2a^2, \quad (73)$$

$${}^{(3)}R_{12} = {}^{(3)}R_{21} = Na, \quad (74)$$

$${}^{(3)}R_{13} = {}^{(3)}R_{31} = 0, \quad (75)$$

$${}^{(3)}R_{22} = -\left(2a^2 + \frac{1}{2}N^2\right), \quad (76)$$

$${}^{(3)}R_{23} = {}^{(3)}R_{32} = 0, \quad (77)$$

$${}^{(3)}R_{33} = -\left(2a^2 + \frac{1}{2}N^2\right). \quad (78)$$

The Ricci scalar is computed from Eqs. (63) and (20) as

$$R = {}^{(3)}R_a^a = -6a^2 - \frac{1}{2}N^2. \quad (79)$$

From Eqs.(61), (73)-(78), and (79), we see that the diagonal components of the shear propagation equations are:

$$\dot{\sigma}_+ + \sqrt{3}\dot{\sigma}_- + \theta\left(\sigma_+ + \sqrt{3}\sigma_-\right) - 72a^2\sigma_+^2 + \frac{2N^2}{3} = 2\eta\kappa\left(\sigma_+ + \sqrt{3}\sigma_-\right), \quad (80)$$

$$\dot{\sigma}_+ - \sqrt{3}\dot{\sigma}_- + \theta\left(\sigma_+ - \sqrt{3}\sigma_-\right) + 72a^2\sigma_+^2 - \frac{N^2}{3} = 2\eta\kappa\left(\sigma_+ - \sqrt{3}\sigma_-\right), \quad (81)$$

and

$$\dot{\sigma}_+ + \sigma_+\theta + \frac{N^2}{6} = 2\eta\kappa\sigma_+. \quad (82)$$

The off-diagonal component of the shear propagation equations is evidently

$$6\dot{a}\sigma_+ + 6a\dot{\sigma}_+ - \frac{6a\sigma_+\dot{N}}{N} + 6\theta a\sigma_+ + 12\sqrt{3}a\sigma_+\sigma_- + N^2a = 12\eta\kappa a\sigma_+. \quad (83)$$

Substituting Eqs. (71) and (70) into Eq. (83), we obtain upon simplification

$$N^2 + 6\dot{\sigma}_+ + 2\sigma_+\theta = 12\kappa\eta\sigma_+. \quad (84)$$

Subtracting Eq. (80) from Eq. (81), we obtain

$$N^4 - 144a^2\sigma_+^2 + 2\sqrt{3}N^2[\dot{\sigma}_- + \sigma_-(\theta - 2\kappa\eta)] = 0. \quad (85)$$

Adding Eqs. (82) and (84), we obtain

$$\frac{-5N^2}{6} - 5\dot{\sigma}_+ - \sigma_+\theta = -10\kappa\eta\sigma_+. \quad (86)$$

One can see that Eqs. (85) and (86) constitute the dynamical equations for the shear variables σ_+ and σ_- .

Substituting Eq. (79) into Eq. (60) we obtain the appropriate form of the generalized Friedmann equation as

$$\sigma^2 = \frac{1}{3}\theta^2 - 3a^2 - \frac{1}{4}N^2 - \kappa\mu. \quad (87)$$

We can use Eq. (87) to rewrite the dynamical equation for the fluid density (Eq. (40)) as

$$\dot{\mu} = -12a^2\eta - 4\kappa\mu\eta - \eta N^2 - \frac{4\mu\theta}{3} + \frac{4\eta\theta^2}{3} + \theta^2\xi, \quad (88)$$

and the Raychaudhuri equation as

$$\dot{\theta} = \frac{1}{2} (12a^2 + 2\kappa\mu + N^2 - 2\theta^2 + 3\kappa\theta\xi). \quad (89)$$

Therefore, one can now see that our Bianchi IV viscous fluid model dynamics is governed by Eqs. (89), (88), (85), (86), (70), and (71):

$$\begin{aligned} \dot{\theta} &= \frac{1}{2} (12a^2 + 2\kappa\mu + N^2 - 2\theta^2 + 3\kappa\theta\xi) \\ \dot{\mu} &= -12a^2\eta - 4\kappa\mu\eta - \eta N^2 - \frac{4\mu\theta}{3} + \frac{4\eta\theta^2}{3} + \theta^2\xi \\ \dot{\sigma}_- &= -\frac{N^2}{2\sqrt{3}} + \frac{24\sqrt{3}a^2\sigma_+^2}{N^2} - \sigma_-\theta + 2\kappa\sigma_-\eta \\ \dot{\sigma}_+ &= -\frac{N^2}{6} - \frac{\sigma_+\theta}{5} + 2\kappa\eta\sigma_+ \\ \dot{a} &= -a \left(\frac{\theta}{3} - 2\sigma_+ \right) \\ \dot{N} &= N \left[\frac{\theta}{3} + 2(\sigma_+ + \sqrt{3}\sigma_-) \right]. \end{aligned} \quad (90)$$

This is a coupled first-order system of six nonlinear ordinary differential equations with free parameters being the shear and bulk viscosity coefficients. Clearly, the system (Eqs. (90)) has no exact solution, so numerical methods must be applied. However, because of the high degree of nonlinearity of the system, numerical algorithms can be difficult to employ to obtain any relevant solutions. Therefore, we will write these equations in the expansion-normalized form [13] [12] which will reduce the number of equations to five.

VI. DYNAMICAL EQUATIONS IN EXPANSION-NORMALIZED VARIABLES

The basic idea of the expansion-normalized variables is that the system (Eqs. (90)) is of the form

$$\frac{d\mathbf{x}}{dt} = f(\mathbf{x}), \quad (91)$$

where $\mathbf{x} = (\theta, \mu, \sigma_-, \sigma_+, a, N) \in \mathbf{R}^6$. We will define $\theta \equiv 3H$, where H is denoted the Hubble scalar and is defined as

$$H = \frac{\dot{s}}{s}, \quad (92)$$

and s is a cosmological length-scale function. It is also convenient to define the cosmological deceleration parameter q

$$q = -\frac{\ddot{s}s}{\dot{s}^2}. \quad (93)$$

Clearly, we have the relationship

$$\dot{H} = -(1 + q)H^2. \quad (94)$$

It is also necessary to introduce a dimensionless time variable τ as

$$s = s_0 e^\tau. \quad (95)$$

From these equations, one can show that

$$\frac{dt}{d\tau} = \frac{1}{H} \Rightarrow \frac{dH}{d\tau} \equiv H' = -(1 + q)H. \quad (96)$$

Substituting $\theta = 3H$ into the Raychaudhuri equation (Eq. (89)), we obtain

$$\dot{H} = -\frac{1}{3}\kappa\mu + \kappa\xi H - \frac{2}{3}\sigma^2 - H^2. \quad (97)$$

We will in addition define: a density parameter: $\Omega \equiv \frac{\mu}{3H^2}$, a scalar shear parameter: $\Sigma^2 \equiv \frac{\sigma^2}{3H^2}$, a curvature parameter: $K \equiv -\frac{{}^{(3)}R}{6H^2}$, curvature parameters: $n \equiv N/H$, $A \equiv a/H$, and shear parameters: $\Sigma_{\pm} \equiv \sigma_{\pm}/H$.

Comparing Eq. (94) and Eq. (97), we immediately see that

$$q = \kappa\Omega - \frac{\kappa\xi}{H} - 2\Sigma^2. \quad (98)$$

At this point we assume that the bulk viscosity coefficient obeys the equation of state

$$\frac{\xi}{H} = 3\xi_0, \quad (99)$$

where ξ_0 is a nonnegative real number, and, Eq. (98) now becomes

$$q = \kappa\Omega - 3\kappa\xi_0 - 2\Sigma^2. \quad (100)$$

We further proceed by writing Eqs. (85), (86), (70), and (71) in expansion-normalized form using the aforementioned definitions of the shear and curvature parameters. Beginning with Eq. (85), we see that

$$\dot{\Sigma}_- = -\frac{n^2 H}{2\sqrt{3}} + 24\sqrt{3}\frac{A^2 \Sigma_+^2 H}{n^2} - 3\Sigma_- H + 2\kappa\Sigma_- \eta - \frac{\dot{H}}{H}\Sigma_-. \quad (101)$$

Using the chain rule, we obtain from Eq. (96) that

$$\frac{d\Sigma_-}{d\tau} = \frac{d\Sigma_-}{dt} \frac{dt}{d\tau} = \dot{\Sigma}_- \frac{1}{H}. \quad (102)$$

Substituting Eq. (102) into Eq. (101), we get that

$$\Sigma'_- = -\frac{n^2}{2\sqrt{3}} + 24\sqrt{3}\frac{A^2 \Sigma_+^2}{n^2} - 3\Sigma_- - 6\kappa\Sigma_- \eta_0 + (1+q)\Sigma_-. \quad (103)$$

As is done by Saha and Rikhvitsky, [17], we have set in Eq. (103), $\eta/H = 3\eta_0$, where η_0 is a nonnegative real number.

Continuing in a similar way with Eq. (86), we have that

$$\Sigma'_+ = -\frac{1}{6}n^2 - \frac{3}{5}\Sigma_+ + 6\kappa\eta_0\Sigma_+ + (1+q)\Sigma_+. \quad (104)$$

In addition, the expansion-normalized forms of Eqs. (71) and (70) become

$$n' = n + 2n \left(\Sigma_+ + \sqrt{3}\Sigma_- \right) + (1+q)n, \quad (105)$$

and

$$A' = -A + 2\Sigma_+ A + (1+q)A. \quad (106)$$

Note that we calculate $\Sigma^2 = \frac{\sigma^2}{3H^2}$ as

$$\begin{aligned} \Sigma^2 &= \frac{1}{6H^2} (\sigma_{ab}\sigma^{ab}) \\ &= \frac{1}{6H^2} (\sigma_{11}^2 + \sigma_{22}^2 + \sigma_{33}^2) \\ &= \frac{1}{6H^2} (\sigma_+^2 + 3\sigma_-^2 + \sigma_+^2 + 3\sigma_-^2 + 4\sigma_+^2) \\ &= \frac{1}{H^2} (\sigma_+^2 + \sigma_-^2) \\ &= \Sigma_+^2 + \Sigma_-^2. \end{aligned} \quad (107)$$

In addition, $K \equiv -\frac{{}^{(3)}R}{6H^2}$ is computed to be

$$\begin{aligned} K &= -\frac{{}^{(3)}R}{6H^2} \\ &= \frac{1}{6H^2} \left(6a^2 + \frac{1}{2}N^2 \right). \\ &= A^2 + \frac{n^2}{12} \end{aligned} \quad (108)$$

The Ω term in the expansion-normalized equations is governed by the expansion-normalized version of Eq. (88), which turns out to be

$$\Omega' = -12A^2\eta_0 - 12\kappa\Omega\eta_0 - \eta_0 n^2 - 4\Omega + 12\eta_0 + 9\xi_0 + 2\Omega(1+q). \quad (109)$$

It is important to note that on physical grounds, $\Omega \geq 0$, which acts as a constraint on the dynamical system. This is because we assume that there is no *exotic* matter in our universe model, that is, the matter content is always assumed to have a nonnegative energy density.

In summary, the expansion-normalized dynamical system is governed by Eqs. (103), (104), (105), (106), and (109):

$$\begin{aligned} \Sigma'_- &= -\frac{n^2}{2\sqrt{3}} + 24\sqrt{3}\frac{A^2\Sigma_+^2}{n^2} - 3\Sigma_- - 6\kappa\Sigma_-\eta_0 + (1+q)\Sigma_- \\ \Sigma'_+ &= -\frac{1}{6}n^2 - \frac{3}{5}\Sigma_+ + 6\kappa\eta_0\Sigma_+ + (1+q)\Sigma_+ \\ A' &= -A + 2\Sigma_+A + (1+q)A \\ n' &= n + 2n\left(\Sigma_+ + \sqrt{3}\Sigma_-\right) + (1+q)n \\ \Omega' &= \eta_0(-12A^2 - 12\kappa\Omega - n^2) - 4\Omega + 12\eta_0 + 9\xi_0 + 2\Omega(1+q). \end{aligned} \quad (110)$$

VII. SOLUTIONS TO THE BIANCHI TYPE IV DYNAMICAL EQUATIONS

In this chapter, we present several numerical solutions to the system of equations Eq. (110). Clearly, this system, which is nonlinear and coupled has no exact solutions. Our goal however is to discover values for ξ_0 and η_0 such that the system asymptotically isotropizes, that is, $\Sigma_{\pm} \rightarrow 0$ as $\tau \rightarrow 0$. Such physical characteristics resemble an anisotropic early-universe that over time becomes isotropic. The solutions to this system of equations were obtained by using the ODE23 solver in MATLAB.

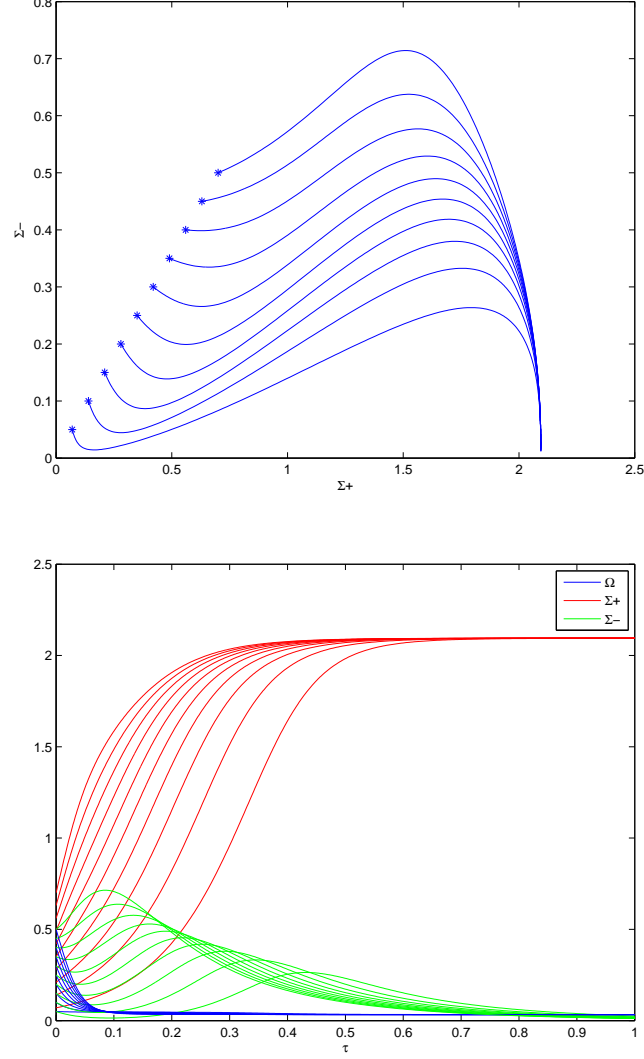
Note that in the diagrams below, asterisks indicate the initial conditions for the numerical experiments. We simulated solutions to the system of equations Eq. (110), by considering ten sets of initial conditions, which had the form

$$[\Sigma_-(0), \Sigma_+(0), A(0), n(0), \Omega(0)] = [0.05i, 0.07i, 0.02i, 0.03i, 0.05i], \quad (111)$$

with i being enumerated from one to ten. Note that there are many possible choices for initial conditions, however, some care must be taken with respect to constraints. In particular, because of the Bianchi IV algebra, one must assure that $A > 0$, $n > 0$, and $\Omega \geq 0$. Clearly the initial conditions chosen satisfy these constraints for all values of i from one through ten.

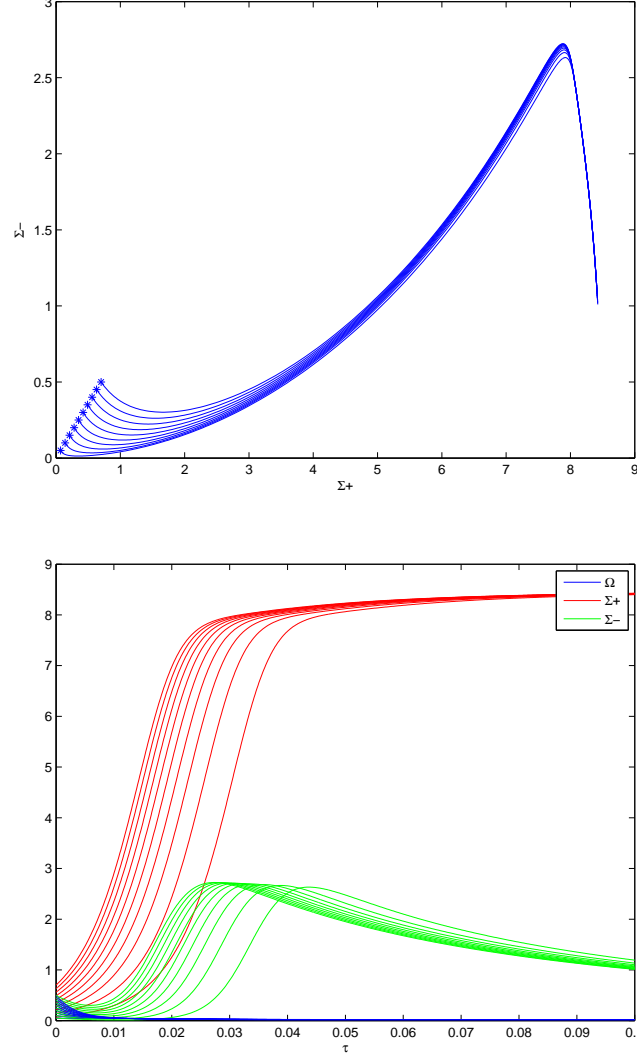
A. $\xi_0 = 0.1, \eta_0 = 0.1$

FIG. 1: The figures below display the dynamical system behaviour for $\xi_0 = 0.1, \eta_0 = 0.1$ in terms of the phase plot of the anisotropy in the Hubble flow, and plots of the energy density and anisotropy variables as functions of time. One can see that $\Omega \rightarrow 0, \Sigma_- \rightarrow 0$, but $\Sigma_+ \not\rightarrow 0$, so the model only partially isotropizes.



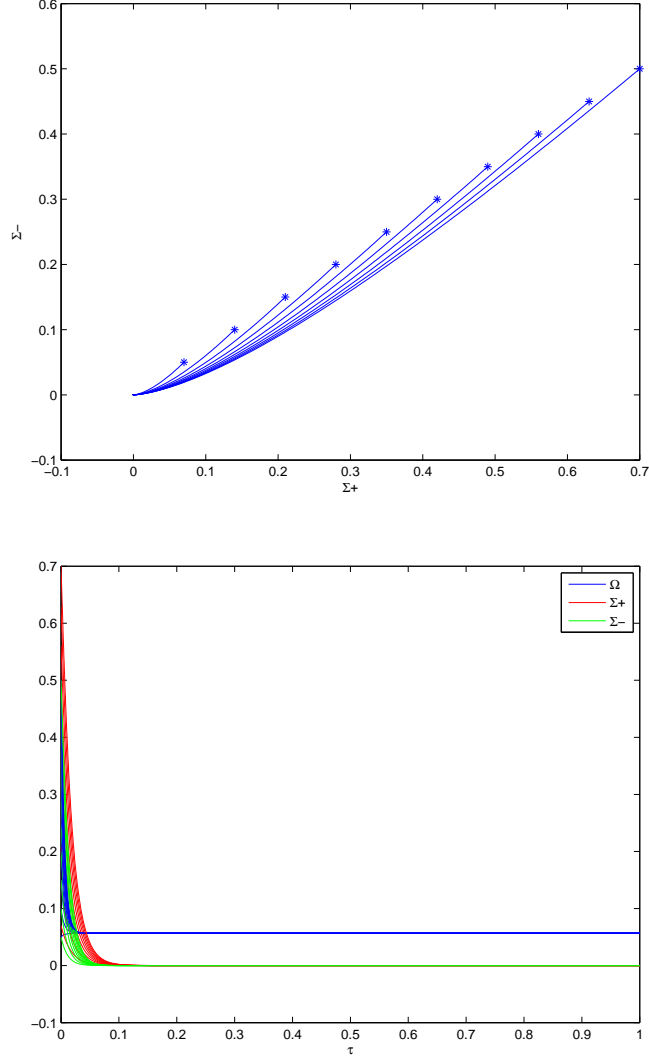
B. $\xi_0 = 0.1, \eta_0 = 1$

FIG. 2: The figures below display the dynamical system behaviour for $\xi_0 = 0.1, \eta_0 = 1$ in terms of the phase plot of the anisotropy in the Hubble flow, and plots of the energy density and anisotropy variables as functions of time. One can see that $\Omega \rightarrow 0, \Sigma_- \rightarrow 0$, but $\Sigma_+ \not\rightarrow 0$, so the model only partially isotropizes.



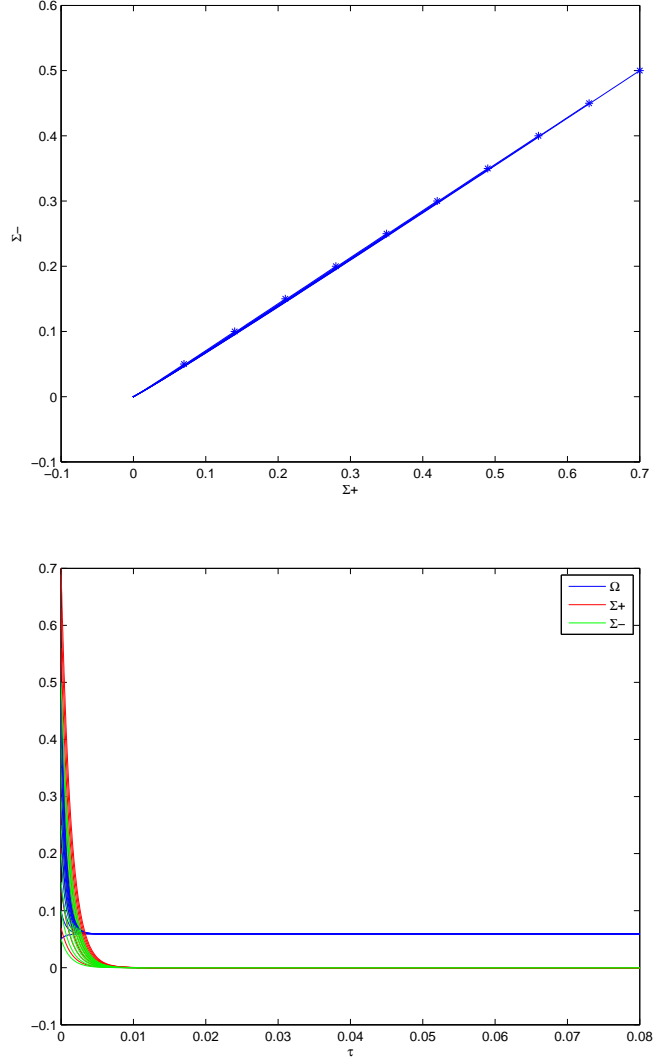
C. $\xi_0 = 1, \eta_0 = 0.1$

FIG. 3: The figures below display the dynamical system behaviour for $\xi_0 = 1, \eta_0 = 0.1$ in terms of the phase plot of the anisotropy in the Hubble flow, and plots of the energy density and anisotropy variables as functions of time. One can see that $\Omega \rightarrow 0, \Sigma_- \rightarrow 0$ and $\Sigma_+ \rightarrow 0$, so the model does in fact isotropize.



D. $\xi_0 = 10, \eta_0 = 0.1$

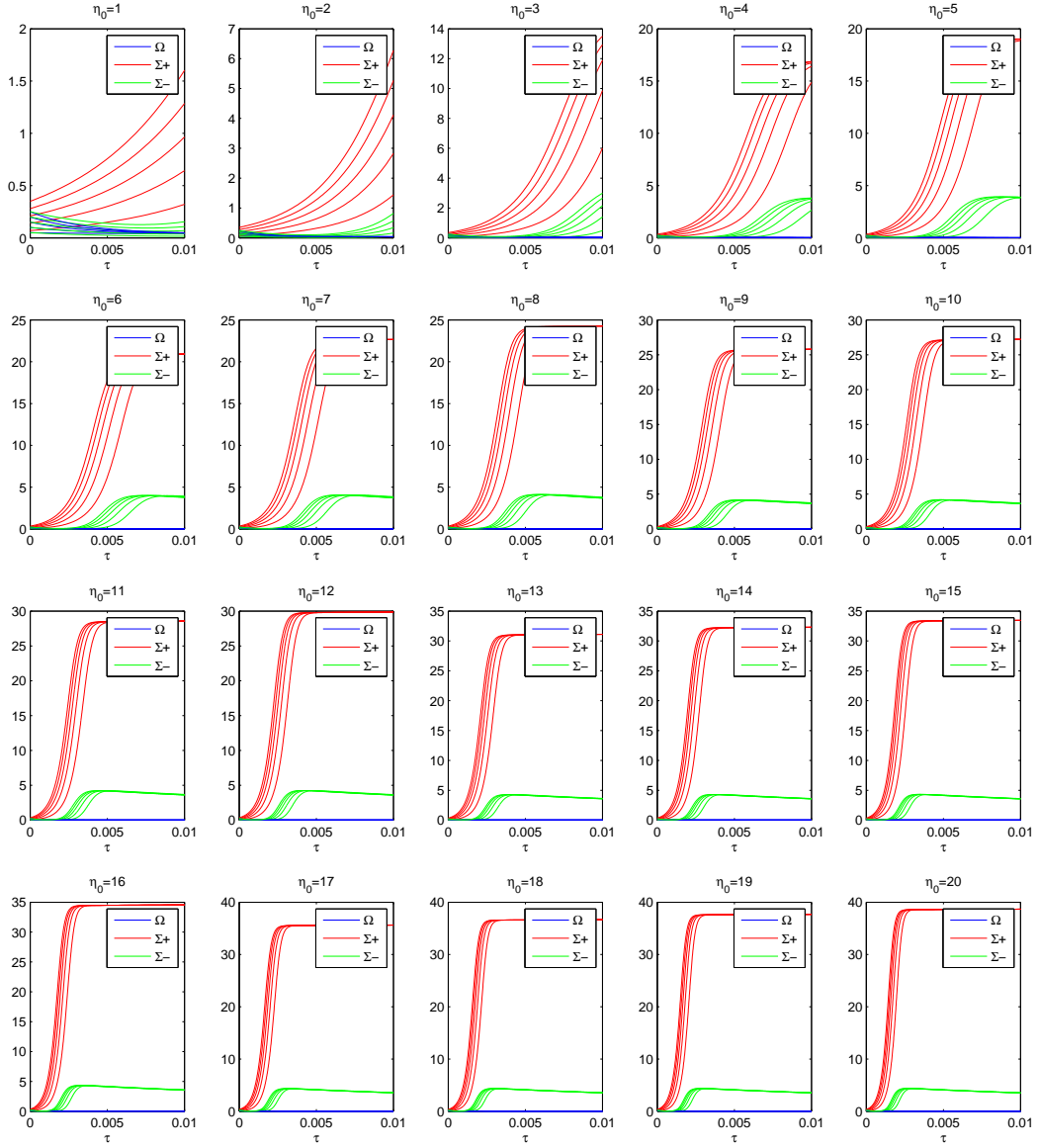
FIG. 4: The figures below display the dynamical system behaviour for $\xi_0 = 10, \eta_0 = 0.1$ in terms of the phase plot of the anisotropy in the Hubble flow, and plots of the energy density and anisotropy variables as functions of time. One can see that $\Omega \rightarrow 0, \Sigma_- \rightarrow 0$ and $\Sigma_+ \rightarrow 0$, so the model does in fact isotropize. (Note that the time scales of the models were adjusted in the figure to display the dynamical behaviour clearly.)



E. Limiting Cases of The Viscosity Coefficients

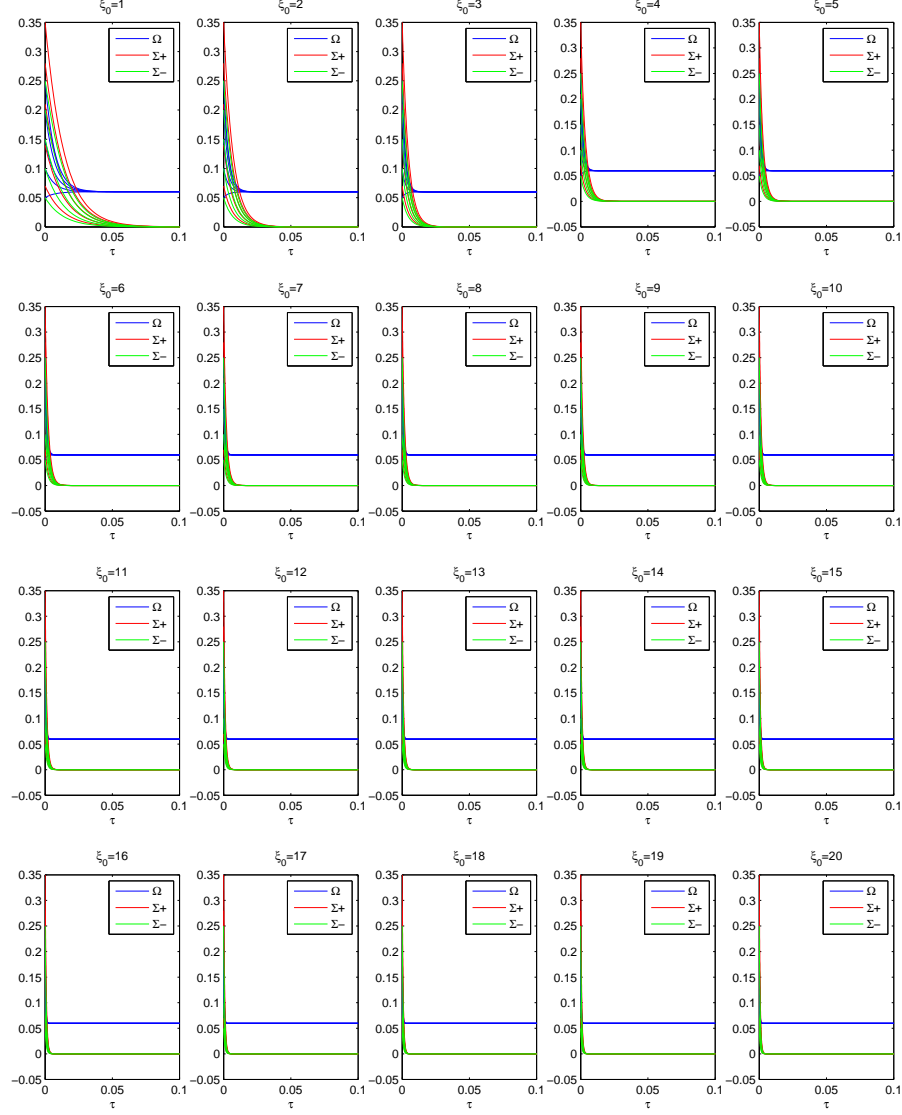
The numerical analysis thus far naturally leads to the question as to under what conditions does this cosmological model isotropize. In an attempt to answer this question, we will take two limiting cases. The first case will be for which the expansion-normalized bulk viscosity coefficient is very small, that is, $\xi_0 \approx 0$ and we will let the expansion-normalized shear viscosity coefficient, η_0 vary. It is therefore appropriate to say in this case that the expansion-normalized shear-viscosity is the dominant viscosity coefficient.

FIG. 5: The different plots in this figure demonstrate the behaviour of the anisotropy functions and energy density as the expansion-normalized shear viscosity coefficient is increased while the expansion-normalized bulk-viscosity coefficient is assumed to be negligible. One sees that although in each case the energy density is bounded such that $\Omega \geq 0$, $\Sigma_{\pm} \not\rightarrow 0$, so the model does not isotropize. In fact, one can see that the anisotropic variables “freeze in” to non-zero asymptotic values after a period of time.



The second case that we consider will be for a negligible expansion-normalized shear viscosity coefficient, that is, $\eta_0 \approx 0$ while the expansion-normalized bulk-viscosity coefficient, ξ_0 is allowed to vary. It is therefore appropriate to say in this case that the expansion-normalized shear-viscosity is the dominant viscosity coefficient.

FIG. 6: The different plots in this figure demonstrate the behaviour of the anisotropy functions and energy density as the expansion-normalized bulk viscosity coefficient is increased while the expansion-normalized shear viscosity coefficient is assumed to be negligible. One sees that not only in each case is the energy density bounded such that $\Omega \geq 0$, but also $\Sigma_{\pm} \rightarrow 0$, so the model does indeed isotropize.



VIII. DISCUSSION OF RESULTS

First looking at Figs. (VII A) and (VII B), one can see that the cosmological model does not isotropize in all directions. In addition, the asymptotic limit of anisotropy in terms of the “freezing-in” of the dynamical variables was also noted by Hervik, van den Hoogen and Coley when studying the Bianchi type IV tilted perfect-fluid model [6]. It is interesting to note that as the shear viscosity coefficient η_0 increased, the anisotropy in the cosmological model increased as well.

However, as can be seen from looking at Figs. (VII C) and (VII D) where the bulk viscosity coefficient ξ_0 was significantly larger than the shear viscosity coefficient, the model isotropized. What is additionally interesting is that as the bulk viscosity was increased, the relationship between the dynamical variables became increasingly linear, which hints at the notion that the dynamics of the transitional period from the early-universe cosmology to the isotropic universe models become simpler. This provides evidence that the presence of a significant bulk viscous pressure rather than a significant shear viscosity is necessary for such a cosmological model to isotropize. These notions were also confirmed in the plots of the limiting cases in Figs. (VII E) and (VII E). In Fig. (VII E), the different plots in this figure demonstrated the behaviour of the anisotropy functions and energy density as the expansion-normalized bulk viscosity coefficient was increased while the expansion-normalized shear viscosity coefficient was assumed to be negligible. One saw that not only in each case was the energy density bounded such that $\Omega \geq 0$, but also $\Sigma_{\pm} \rightarrow 0$, so the model did indeed isotropize. On the other hand, in Fig. (VII E), the different plots demonstrated the behaviour of the anisotropy functions and energy density as the expansion-normalized shear viscosity coefficient was increased while the expansion-normalized bulk-viscosity coefficient was assumed to be negligible. We observed that although in each case the energy density was bounded such that $\Omega \geq 0$, $\Sigma_{\pm} \not\rightarrow 0$, so the model did not isotropize. In fact, we additionally observed that the anisotropic variables “freeze in” to non-zero asymptotic values after a period of time.

Similar results for other Bianchi models have been reported by van den Hoogen and Coley [4], Singh and Kale [18], and Pradhan, Rai and Singh [5]. It is also important to note that no cosmological model that is spatially homogeneous and anisotropic and necessarily has a three-dimensional isometry group can evolve to a model with a six-dimensional isometry group. When one speaks of a model isotropizing, $\Sigma_{\pm} \rightarrow 0$ only asymptotically. However, any Bianchi model can become arbitrary close to that of the FLRW models as a consequence of asymptotic isotropization.

IX. CONCLUSIONS

In this paper, we were interested in formulating a viscous model of the early universe based on The Bianchi Type IV algebra. We first began by considering a congruence of fluid lines in spacetime, upon which, analyzing their propagation behaviour, we rigorously derived the famous Raychaudhuri equation, suitably modified to accommodate viscous fluid effects. We then went through in detail the topological and algebraic structure of a Bianchi Type IV algebra, by which we derived the corresponding structure and constraint equations. From this, we looked at the Einstein field equations in the context of orthonormal frames, and derived and discussed the resulting dynamical equations: The *Raychaudhuri Equation*, *generalized Friedmann equation*, *shear propagation equations*, and a set of non-trivial constraint equations. It is shown that for cases where the expansion-normalized bulk viscosity coefficient ξ_0 dominates the expansion-normalized shear viscosity coefficient, η_0 , the model isotropizes.

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